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ON THE PROPAGATION OF LONG-WAVE PERTURBATIONS IN A TWO-LAYER FREE-BOUNDARY ROTATIONAL FLUID

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A mathematical model for the propagation of long-wave perturbations in a free-boundary shear flow of an ideal stratified two-layer fluid is considered. The characteristic equation defining the velocity of perturbation propagation in the fluid is obtained and studied. The necessary hyperbolicity conditions for the equations of motion are formulated for flows with a monotonic velocity profile over depth, and the characteristic form of the system is calculated. It is shown that the problem of deriving the sufficient hyperbolicity conditions is equivalent to solving a system of singular integral equations. The limiting cases of weak and strong stratification are studied. For these models, the necessary and sufficient hyperbolicity conditions are formulated, and the equations of motion are reduced to the Riemann integral invariants conserved along the characteristics.

Key words: *two-layer fluid, shear flows, long waves, hyperbolicity.*

Introduction. The modeling of nonlinear wave motions on the surface of a narrow fluid layer is of interest for both basic research and applications in oceanology and meteorology. This topic has long been developed and has been the subject of extensive research [1–4]. The study of the dynamics of a stratified fluid with piecewise-constant density has applications to problems of oceanology and research into the stratified (separated into horizontal layers with nearly uniform density) structure of the upper ocean layer [5]. An important feature of such motions is the development of internal waves due to momentum transfer from one layer to another. Ovsyannikov [6] developed and studied models for the potential wave motions of a two-layer fluid in the asymptotic shallow-water approximation. The new theoretical method for analyzing integrodifferential equations proposed by Teshukov [7] allows the study of more complex models of two-layer fluids taking into account the rotational (shear) nature of the motion. A number of results for the two-layer model of rotational shallow water with a free boundary and with a rigid boundary were obtained in [8, 9].

1. Derivation of Mathematical Model. The plane-parallel free-boundary flow of an ideal incompressible two-layer heavy fluid above an even bottom is described by the Euler equations with the appropriate boundary and initial conditions:

$$u_{it} + u_i u_{ix} + v_i u_{iy} + \rho_i^{-1} p_{ix} = 0, \qquad \varepsilon^2 (v_{it} + u_i v_{ix} + v_i v_{iy}) + \rho_i^{-1} p_{iy} = -g,$$

$$u_{ix} + v_{iy} = 0, \qquad h_{1t} + u_1(t,x,h_1) h_{1x} = v_1(t,x,h_1), \qquad v_1(t,x,0) = 0,$$

$$h_{1t} + u_2(t,x,h_1) h_{1x} = v_2(t,x,h_1), \tag{1}$$

$$(h_1 + h_2)_t + u_2(t,x,h_1+h_2)(h_1+h_2)_x = v_2(t,x,h_1+h_2),$$

$$u_i(0,x,y) = u_{i0}(x,y), \quad v_i(0,x,y) = v_{i0}(x,y), \quad h_i(0,x) = h_{i0}(x) \qquad (i = 1,2).$$

The subscripts $i = 1$ and 2 correspond to the hydrodynamic quantities in the lower and upper layers of the fluid, respectively (Fig. 1). The variables $u_i^* = (aH_0)^{1/2} u_i$, $v_i^* = (aH_0)^{1/2} H_0 L_0^{-1} v_i$, $\rho_i^* = R_0 \rho_i$, $p_i^* = R_0 a H_0 p_i$,

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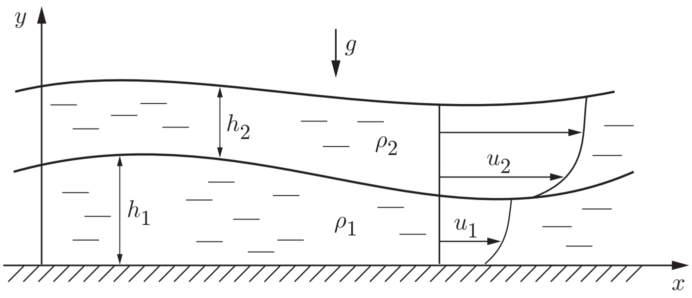

Fig. 1.

$t^* = L_0(aH_0)^{-1/2}t$, $x^* = L_0 x$, and $y^* = H_0 y$ are the dimensional velocity components, density, pressure, time, and Cartesian coordinates, respectively; u_i, v_i, ρ_i, p_i, t, x, and y are their corresponding nondimensional quantities. The parameters L_0 and H_0 determine the characteristic horizontal and vertical scales; the parameter R_0 has the density dimension, a is the acceleration; g is the nondimensional acceleration due to gravity; and $h_1(t,x)$ and $h_2(t,x)$ are the depths of the layers with densities ρ_1 and ρ_2 ($\rho_1 > \rho_2$).

In the long-wave approximation, the dimensionless parameter $\varepsilon = H_0/L_0$ is considered small. Neglecting terms of order ε^2 in Eqs. (1), we obtain the following expressions for the hydrostatic pressure distribution:

$$\begin{aligned} &p_1 = g\rho_1(h_1 - y) + g\rho_2 h_2 + p_0 \qquad (0 \leqslant y \leqslant h_1),\\ &p_2 = g\rho_2(h_1 + h_2 - y) + p_0 \qquad (h_1 \leqslant y \leqslant h_1 + h_2). \end{aligned} \tag{2}$$

Integration of the continuity equation subject to the boundary conditions gives the following expressions for the vertical velocity component in the layers:

$$v_1 = -\int_0^y u_{1x}(t,x,y')\,dy', \qquad v_2 = -\int_{h_1}^y u_{2x}(t,x,y')\,dy' + h_{1t} + u_2(t,x,h_1)h_{1x};$$

system (1) for $\varepsilon = 0$ takes the form

$$\begin{aligned} &u_{1t} + u_1u_{1x} + v_1u_{1y} + gh_{1x} + grh_{2x} = 0, \qquad h_{1t} + \Big(\int_0^{h_1} u_1\,dy\Big)_x = 0,\\ &u_{2t} + u_2u_{2x} + v_2u_{2y} + g(h_{1x} + h_{2x}) = 0, \qquad h_{2t} + \Big(\int_{h_1}^{h_1+h_2} u_2\,dy\Big)_x = 0 \end{aligned} \tag{3}$$

[$r = \rho_2/\rho_1$; the expressions for $v_i(t,x,y)$ are given above and the initial data are the same]. It should be noted that in the approximation considered, the vorticity in the layer is proportional to u_{iy}, and in the case of no velocity shear over the depth, system (3) reduces to the well-known equations of two-layer shallow water [6]. We consider flows with a monotonic velocity profile over the depth. For definiteness, let $u_{iy} > 0$ and $u_1 < u_2$.

The characteristic properties of the two-layer shallow-water equations for shear flows (3) are conveniently analyzed in a semi-Lagrangian coordinate system using Teshukov's generalization [4, 7] of the characteristic and hyperbolicity concepts for systems with operator coefficients. Conversion to the semi-Lagrangian variables x and λ ($0 \leqslant \lambda \leqslant 1$) is performed by the substitution of variables [10]

$$y = \begin{cases} \Phi_1(t,x,\lambda), & \text{if } 0 \leqslant y \leqslant h_1,\\ h_1(t,x) + \Phi_2(t,x,\lambda), & \text{if } h_1 < y \leqslant h_1 + h_2. \end{cases} \tag{4}$$

The functions $\Phi_i(t,x,\lambda)$ are the solutions of the following Cauchy problems:

$$\Phi_{1t} + u_1(t,x,\Phi_1)\Phi_{1x} = v_1(t,x,\Phi_1), \qquad \Phi_1|_{t=0} = \lambda h_1(0,x);$$

$$(h_1 + \Phi_2)_t + u_2(t,x,h_1+\Phi_2)(h_1+\Phi_2)_x = v_2(t,x,h_1+\Phi_2), \quad (h_1+\Phi_2)|_{t=0} = h_1(0,x) + \lambda h_2(0,x).$$

The substitution is inversible if $\Phi_{i\lambda} \neq 0$ (we set $\Phi_{i\lambda} > 0$).

The functions $u_i(t,x,\lambda)$ and $H_i(t,x,\lambda) = \Phi_{i\lambda}$ are obtained by solving the integrodifferential equations

$$\begin{aligned} &u_{1t} + u_1 u_{1x} + g\int_0^1 H_{1x}\,d\lambda + gr\int_0^1 H_{2x}\,d\lambda = 0, \qquad H_{1t} + H_1 u_{1x} + u_1 H_{1x} = 0, \\ &u_{2t} + u_2 u_{2x} + g\int_0^1 H_{1x}\,d\lambda + g\int_0^1 H_{2x}\,d\lambda = 0, \qquad H_{2t} + H_2 u_{2x} + u_2 H_{2x} = 0 \end{aligned} \tag{5}$$

subject to the initial data $u_i(0,x,\lambda) = u_{0i}(x,\lambda)$ and $H_i(0,x,\lambda) = H_{0i}(x,\lambda)$.

Once the functions $u_i(t,x,\lambda)$ and $H_i(t,x,\lambda)$ are found from the solution of Eqs. (5), the formulas $\Phi_{i\lambda} = H_i$ and $\Phi_i(t,x,0) = 0$ allow one to find Φ_i and the depth of layers $h_i = \Phi_i(t,x,1)$. The pressure is determined from formulas (2) and (4), and the vertical velocity components are obtained from the expressions for v_i. Next, we find the hyperbolicity conditions of the equations of two-layer rotational shallow water (5).

2. Characteristic Properties of Eqs. (5). System (5) is written as

$$\boldsymbol{U}_t + A\boldsymbol{U}_x = 0, \tag{6}$$

where $\boldsymbol{U} = (u_1, H_1, u_2, H_2)^{\rm t}$ is the desired vector;

$$A = \begin{pmatrix} u_1 & g\int_0^1 \ldots\,d\lambda & 0 & gr\int_0^1 \ldots\,d\lambda \\ H_1 & u_1 & 0 & 0 \\ 0 & g\int_0^1 \ldots\,d\lambda & u_2 & g\int_0^1 \ldots\,d\lambda \\ 0 & 0 & H_2 & u_2 \end{pmatrix}$$

is a matrix with operator coefficients.

According to [7], the characteristic curve of system (6) is defined by the differential equation $x'(t) = k(t,x)$, where the rate of propagation of the characteristic k is an eigenvalue of the problem

$$(\boldsymbol{F}, (A - kI)\boldsymbol{\varphi}) = 0. \tag{7}$$

The solution of Eq. (7) for the vector functional $\boldsymbol{F} = (F_1, F_2, F_3, F_4)$ is sought in the class of locally integrable or generalized functions. The functional $\boldsymbol{F}$ acts on functions of the variable λ (t and x are treated as parameters) and I is an identical mapping. The action of the functional $\boldsymbol{F}$ on Eq. (6) yields the characteristic relation

$$(\boldsymbol{F}, \boldsymbol{U}_t + k\boldsymbol{U}_x) = 0. \tag{8}$$

System (6) is a generalized-hyperbolic [7] if all eigenvalues of k are real and the set of relations on the characteristics (8) is equivalent to Eqs. (6) (i.e., the system of eigenfunctionals is complete in the space considered).

Taking into account the independence of the components of the trial vector function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^{\rm t}$, from Eqs. (7) we obtain the equalities

$$\begin{aligned} &(F_1, (u_1 - k)\varphi_1) + (F_2, H_1\varphi_1) = 0, \qquad g\int_0^1 \varphi_2\,d\lambda (F_1, 1) + (F_2, (u_1 - k)\varphi_2) + g\int_0^1 \varphi_2\,d\lambda (F_3, 1) = 0, \\ &(F_3, (u_2 - k)\varphi_3) + (F_4, H_2\varphi_3) = 0, \qquad gr\int_0^1 \varphi_4\,d\lambda (F_1, 1) + g\int_0^1 \varphi_4\,d\lambda (F_3, 1) + (F_4, (u_2 - k)\varphi_4) = 0. \end{aligned} \tag{9}$$

Let us consider the set of numbers k belonging to the complex plane, except for the segments $[u_{10}, u_{11}]$ and $[u_{20}, u_{21}]$, where $u_{i0} = u_i(t,x,0)$ and $u_{i1} = u_i(t,x,1)$. From system (9) it follows that

$$(F_1, \psi_1) = -(F_2, (u_1 - k)^{-1} H_1 \psi_1), \qquad (F_3, \psi_3) = -(F_4, (u_2 - k)^{-1} H_2 \psi_3), \tag{10}$$

where $\psi_{1,2} = (u_1 - k)\varphi_{1,2}$ and $\psi_{3,4} = (u_2 - k)\varphi_{3,4}$. Therefore, the action of the components F_2 and F_4 of the vector functional $\boldsymbol{F}$ is written as

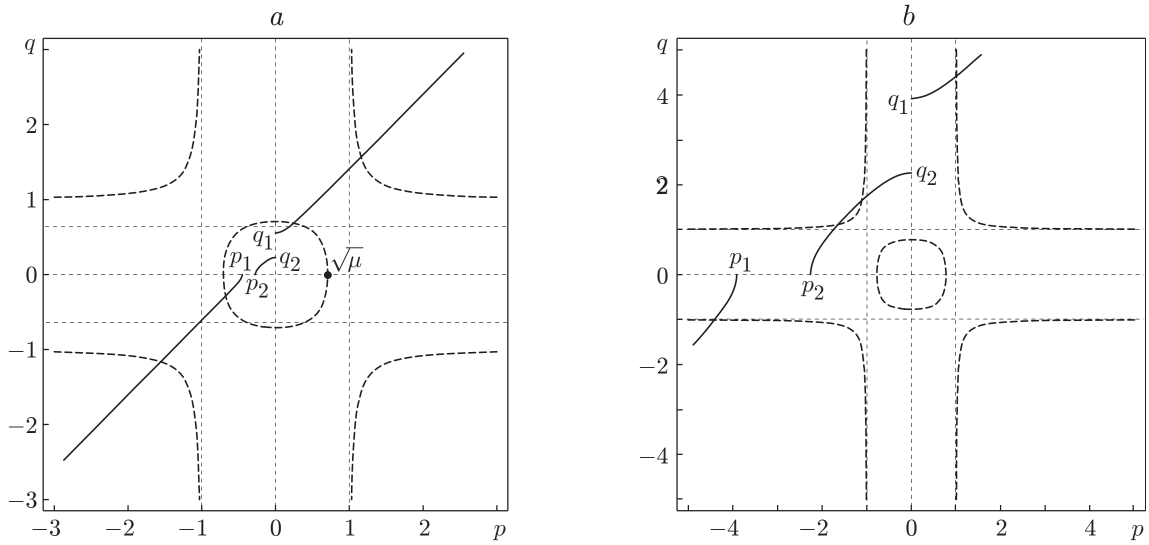

Fig. 2.

$$(F_2, \psi_2) = g[(F_2, (u_1 - k)^{-1} H_1) + (F_4, (u_2 - k)^{-1} H_2)] \int_0^1 (u_1 - k)^{-1} \psi_2 \, d\lambda,$$
$$(F_4, \psi_4) = g[r(F_2, (u_2 - k)^{-1} H_1) + (F_4, (u_2 - k)^{-1} H_2)] \int_0^1 (u_1 - k)^{-1} \psi_4 \, d\lambda. \quad (11)$$

Setting $\psi_2 = (u_1 - k)^{-1} H_1$ and $\psi_4 = (u_2 - k)^{-1} H_2$ in formulas (11) and assuming zero value for the corresponding determinant of the system homogeneous in $(F_2, (u_1 - k)^{-1} H_1)$ and $(F_4, (u_2 - k)^{-1} H_2)$, we obtain the characteristic equation

$$\chi(k) = 1 - g \int_0^1 \frac{H_1 \, d\lambda}{(u_1 - k)^2} - g \int_0^1 \frac{H_2 \, d\lambda}{(u_2 - k)^2} + g^2 \mu \int_0^1 \frac{H_1 \, d\lambda}{(u_1 - k)^2} \int_0^1 \frac{H_2 \, d\lambda}{(u_2 - k)^2} = 0, \quad (12)$$

where $\mu = 1 - r$ $(0 < \mu < 1)$.

As shown in [6], in the irrotational case, the two-layer shallow-water equation are hyperbolic for the solution considered if the characteristic equation have four real roots. In this case, therefore, the most natural situation is the one where for the examined solution of the integrodifferential system (5) there are also four real characteristic roots k_i. In view of the complexity and nonlinearity of Eq. (12), it is conveniently analyzed using the geometrical interpretation proposed in [6] for the averaged model. We designate

$$\frac{1}{p^2(k)} = g \int_0^1 \frac{H_1 \, d\lambda}{(u_1 - k)^2}, \qquad \frac{1}{q^2(k)} = g \int_0^1 \frac{H_2 \, d\lambda}{(u_2 - k)^2} \quad (13)$$

[the signs of the quantities $p(k)$ and $q(k)$ coincide with the signs of the quantities $u_1 - k$ and $u_2 - k$, respectively; $k \in (-\infty, u_{10}) \cup (u_{11}, u_{20}) \cup (u_{21}, \infty)$]. Then, Eqs. (12) is written as

$$(p^2 - 1)(q^2 - 1) = r. \quad (14)$$

On the plane of the dimensionless variables (p, q), Eq. (14) is the equation of a fourth-order curve with four symmetry axis (dashed curve in Fig. 2). The number of the real roots of Eq. (12) is determined by the number of intersections of the curve (14) with the parametrically defined discontinuous curve (13) (solid curve in Fig. 2). In the irrotational case, formulas (13) define a straight line on the plane (p, q). The number of real solutions of Eq. (12) is conveniently determined with the use of the quantities

$$p_1 = -\Big(g \int_0^1 \frac{H_1\,d\lambda}{(u_1 - u_{21})^2}\Big)^{-1/2}, \qquad p_2 = -\Big(g \int_0^1 \frac{H_1\,d\lambda}{(u_1 - u_{20})^2}\Big)^{-1/2},$$

$$q_1 = \Big(g \int_0^1 \frac{H_2\,d\lambda}{(u_2 - u_{10})^2}\Big)^{-1/2}, \quad q_2 = \Big(g \int_0^1 \frac{H_2\,d\lambda}{(u_2 - u_{11})^2}\Big)^{-1/2}.$$

We note that $p_1 < p_2 < 0$, $0 < q_2 < q_1$, and the parametrically defined curve $p = p(k)$, $q = q(k)$ increases monotonically on the segments $(-\infty, u_{10})$, (u_{11}, u_{20}), and (u_{21}, ∞). We formulate the sufficient conditions for the existence of four real roots of Eq. (12).

If for a given solution u_i, H_i, one of the following conditions is satisfied:

$$\begin{aligned} &1)\quad q_1 < \sqrt{\mu}, \quad p_1 > -\sqrt{\mu};\\ &2)\quad q_2 > \sqrt{\mu}, \quad p_1 > -\sqrt{\mu};\\ &3)\quad q_1 < \sqrt{\mu}, \quad p_2 < -\sqrt{\mu};\\ &4)\quad q_2 > \sqrt{\mu}, \quad p_2 < -\sqrt{\mu}, \quad p^2(k_*) + q^2(k_*) \leqslant \mu,\\ &5)\quad p^2(k_*) > 1 + \sqrt{r}, \quad q^2(k_*) > 1 + \sqrt{r} \quad (k_* = (u_{20} + u_{11})/2), \end{aligned} \tag{15}$$

then Eq. (12) *has four real roots.*

Each of conditions in (15) guarantees the existence of four points of intersection of curves (13) and (14) that correspond to the real roots of the characteristic equation (12). Figure 2a corresponds to condition 1, and Fig. 2b corresponds to condition 5. If condition 4 or 5 is satisfied, the following order of the characteristic roots takes place: $k_1 < u_1 < k_2 < k_3 < u_2 < k_4$. For condition 1, the inequalities $k_1 < k_2 < u_1 < u_2 < k_3 < k_4$ are valid; if condition 2 (or 3) is satisfied, there is one root on the segment (u_{11}, u_{20}), one (or two) root(s) on the segment $(-\infty, u_{10})$, and two (or one) root on the segment (u_{21}, ∞).

In the case of potential flows, where the characteristic equation is a fourth-order polynomial, the use of the geometrical interpretation given above is sufficient to elucidate the type of equations. For rotational flows, Eq. (12), which defines the characteristic roots, is not a polynomial and the conditions for the absence of complex roots are more complex.

Let us define the complex functions $\chi_i(z)$ by

$$\chi_i(z) = \frac{1}{\omega_{i1}(u_{i1} - z)} - \frac{1}{\omega_{i0}(u_{i0} - z)} - \int_0^1 \frac{\partial}{\partial \nu}\Big(\frac{1}{\omega_i'}\Big)\frac{d\nu}{u_i' - z},$$

where $\omega_i = u_{i\lambda}/H_i$ and $u_i' = u_i(t, x, \nu)$. Then, the function $\chi(z)$ is written as

$$\chi(z) = 1 + g\chi_1(z) + g\chi_2(z) + g^2\mu\chi_1(z)\chi_2(z).$$

The conditions for the absence of complex roots of Eq. (12) are formulated in terms of the limiting values of the function $\chi(z)$ from the upper and lower half-planes on the real axis.

Lemma 1. *For the solution $u_i(t, x, \lambda)$, $H_i(t, x, \lambda)$, Eq.* (12) *does not have complex roots if the condition*

$$\Delta\arg(\chi^+(u)/\chi^-(u)) = 2\pi(n - 4) \qquad (\chi^\pm \neq 0) \tag{16}$$

is satisfied (*the argument increment is calculated for u varied from u_{10} to u_{11} and from u_{20} to u_{21}; n is the number of real zeroes of the function χ*).

Lemma 1 is proved by applying the argument principle to the analytic function $\chi(z)$, as was done in [11] for the barotropic single-layer model.

In [8], an n-layer model was considered and a characteristic equation was derived, which, for $n = 2$, coincides with Eq. (12). The same paper gives some conditions for the existence of four real roots for the two-layer model in terms of the roots of auxiliary functions of the form $1 + \mu g\chi_i$ and $1 + (1 + \sqrt{r})g\chi_i$. There are two significant differences between the conditions given in [8] and conditions in (15) proposed in the present paper. The cases where on the segment (u_{11}, u_{20}) there is one characteristic root but the total number of real roots is four [which corresponds to condition 3 or 4 in (15)] are not considered in [8]. For the case where on the segment (u_{11}, u_{20}) there are two roots but the total number of roots is equal to four, only one condition [the analog of condition 5 corresponding to large values of the Froude number] is given. The other situation where on the solution considered there are four roots (12), two of which are on the segment (u_{11}, u_{20}) [condition 4 in (15)] is not considered in [8].

Eigenfunctionals. Using formulas (10) and (11), it is easy to find eigenfunctionals that correspond to the roots $k = k_i$ of the characteristic equations (12):

$$(\boldsymbol{F}^i, \boldsymbol{\varphi}(\lambda)) = -(1-\gamma_2^i)\Big(\int_0^1 \frac{H_1\varphi_1\,d\lambda}{(u_1-k_i)^2} - \int_0^1 \frac{\varphi_2\,d\lambda}{(u_1-k_i)^2}\Big) - r\gamma_1^i\Big(\int_0^1 \frac{H_2\varphi_3\,d\lambda}{u_2-k_i} - \int_0^1 \frac{\varphi_4\,d\lambda}{u_2-k_i}\Big),$$

$$\gamma_j^i = g\int_0^1 \frac{H_j\,d\nu}{(u_j-k_i)^2} \qquad (j=1,2)$$

[by virtue of (12), the equality $1-\gamma_1^i-\gamma_2^i+\mu\gamma_1^i\gamma_2^i = 0$ is valid].

Let us show that problem (7) has nontrivial solutions if $k \in [u_{10}, u_{12}] \cup [u_{20}, u_{21}]$, i.e., if there is a continuous characteristic spectrum consisting of two segments of the real axis. Let $k = u_1(t,x,\lambda)$. In this case, system (9) becomes

$$\begin{aligned}&(F_1, (u_1'-u_1)\varphi_1') + (F_2, H_1'\varphi_1') = 0, \qquad g\int_0^1 \varphi_2\,d\lambda (F_1,1) + (F_2,(u_1'-u_1)\varphi_2') + g\int_0^1 \varphi_2\,d\lambda(F_3,1) = 0,\\ &(F_3, (u_2'-u_1)\varphi_3') + (F_4, H_2'\varphi_3') = 0, \qquad gr\int_0^1 \varphi_4\,d\lambda(F_1,1) + g\int_0^1 \varphi_4\,d\lambda(F_3,1) + (F_4,(u_2'-u_1)\varphi_4') = 0.\end{aligned} \tag{17}$$

Here the functionals act over the variable ν; the notation $f' = f(t,x,\nu)$, $f = f(t,x,\lambda)$ is used for brevity. Since $H_i \neq 0$, the equalities

$$(F_2, \psi') = -(F_1, (u_1'-u_1){H'}_1^{-1}\psi'), \qquad (F_4,\psi') = -(F_3,(u_2'-u_1){H'}_2^{-1}\psi')$$

hold and system (17) reduces to the equations

$$g\int_0^1 \varphi_2\,d\lambda(F_1,1) + g\int_0^1 \varphi_2\,d\lambda(F_3,1) - (F_1,(u_1'-u_1)^2{H'}_1^{-1}\varphi_2') = 0,$$

$$gr\int_0^1 \varphi_4\,d\lambda(F_1,1) + g\int_0^1 \varphi_4\,d\lambda(F_3,1) - (F_3,(u_2'-u_1)^2{H'}_2^{-1}\varphi_4') = 0.$$

This system has two different solutions $\boldsymbol{F}^{1\lambda}$ and $\boldsymbol{F}^{2\lambda}$:

$$(\boldsymbol{F}^{1\lambda}, \boldsymbol{\varphi}(\nu)) = (1-\alpha\mu)g\int_0^1 \frac{H_1'(\varphi_1'-\varphi_1)\,d\nu}{(u_1'-u_1)^2} + (1-\alpha)\varphi_1(\lambda)$$

$$-(1-\alpha\mu)g\int_0^1 \frac{\varphi_2'\,d\nu}{u_1'-u_1} + rg\int_0^1 \frac{H_2'\varphi_3'\,d\nu}{(u_2'-u_1)^2} - rg\int_0^1 \frac{\varphi_4'\,d\nu}{u_2'-u_1},$$

$$(\boldsymbol{F}^{2\lambda}, \boldsymbol{\varphi}(\nu)) = -\varphi_{1\lambda} + u_{1\lambda}H_1^{-1}\varphi_2(\lambda).$$

Similarly, we find the eigenfunctionals $\boldsymbol{F}^{3\lambda}$ and $\boldsymbol{F}^{4\lambda}$ corresponding to the values $k = u_2(t,x,\lambda)$:

$$(\boldsymbol{F}^{3\lambda}, \boldsymbol{\varphi}(\nu)) = g\int_0^1 \frac{H_1'\varphi_1'\,d\nu}{(u_1'-u_2)^2} - g\int_0^1 \frac{\varphi_2'\,d\nu}{(u_1'-u_2)^2} + (1-\beta)\varphi_3(\lambda) + (1-\beta\mu)g\int_0^1 \frac{H_2'(\varphi_3'-\varphi_3)\,d\nu}{u_2'-u_2} - (1-\beta\mu)g\int_0^1 \frac{\varphi_4'\,d\nu}{u_2'-u_2},$$

$$(\boldsymbol{F}^{4\lambda}, \boldsymbol{\varphi}(\nu)) = -\varphi_{3\lambda} + u_{2\lambda}H_2^{-1}\varphi_4(\lambda).$$

Here the following notation was used:

$$\alpha = g\int_0^1 \frac{H_2'\,d\nu}{(u_2'-u_1)^2}, \qquad \beta = g\int_0^1 \frac{H_1'\,d\nu}{(u_1'-u_2)^2}.$$

Characteristic Relations. Acting on system (5) by the eigenfunctionals $\boldsymbol{F}^{i\lambda}$ and $\boldsymbol{F}^i$, we obtain the following characteristic relations:

$$\omega_{1t} + u_1\omega_{1x} = 0, \qquad \omega_{2t} + u_2\omega_{2x} = 0,$$

$$\frac{1-\alpha\mu}{rg}\Big(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x}\Big)\Big(u_1 - g\int_0^1 \frac{H_1'\,d\nu}{u_1'-u_1}\Big) - \Big(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x}\Big)\int_0^1 \frac{H_2'\,d\nu}{u_2'-u_1} = 0,$$

$$(1-\beta\mu)\Big(\frac{\partial}{\partial t} + u_2\frac{\partial}{\partial x}\Big)\Big(u_2 - g\int_0^1 \frac{H_2'\,d\nu}{u_2'-u_2}\Big) - r\Big(\frac{\partial}{\partial t} + u_2\frac{\partial}{\partial x}\Big)g\int_0^1 \frac{H_2'\,d\nu}{u_1'-u_2} + \mu\beta\Big(\frac{\partial}{\partial t} + u_2\frac{\partial}{\partial x}\Big)u_2 = 0, \qquad (18)$$

$$\frac{1-\gamma\mu}{rg}\Big(\frac{\partial}{\partial t} + k_i\frac{\partial}{\partial x}\Big)\Big(k_i - g\int_0^1 \frac{H_1\,d\lambda}{u_1-k_i}\Big) - \Big(\frac{\partial}{\partial t} + k_i\frac{\partial}{\partial x}\Big)\int_0^1 \frac{H_2\,d\lambda}{u_2-k_i} = 0.$$

Next, we assume that for the solution considered, one of conditions in (15) is satisfied [i.e., Eq. (12) has four real roots] and the condition of no complex characteristics (16) is satisfied. We consider the problem of the completeness of the system of eigenfunctionals $\boldsymbol{F}^{i\lambda}$ and $\boldsymbol{F}^i$ $(i = 1, 2, 3, 4)$. To prove that Eqs. (5) are equivalent to the characteristic relations (18), it is necessary to show that the equalities $(\boldsymbol{F}^{i\lambda}, \boldsymbol{S}) = 0$ and $(\boldsymbol{F}^i, \boldsymbol{S}) = 0$ are satisfied if and only if the vector function $\boldsymbol{S}(\lambda) = (S_1, S_2, S_3, S_4)$ is identically equal to zero.

From the equations $(\boldsymbol{F}^{2\lambda}, \boldsymbol{S}) = 0$ and $(\boldsymbol{F}^{4\lambda}, \boldsymbol{S}) = 0$ it follows that $S_2 = \omega_1^{-1} S_{1\lambda}$ and $S_4 = \omega_2^{-1} S_{3\lambda}$. Bearing this in mind, we write the results of action of the functionals $\boldsymbol{F}^{1\lambda}$ and $\boldsymbol{F}^{3\lambda}$ on the vector function $\boldsymbol{S}$:

$$\frac{1-\alpha}{rg}S_1 - \frac{1-\mu\alpha}{r}\int_0^1 \frac{1}{\omega_1'}\frac{\partial}{\partial\nu}\Big(\frac{S_1'-S_1}{u_1'-u_1}\Big)d\nu - \int_0^1 \frac{1}{\omega_2'}\frac{\partial}{\partial\nu}\Big(\frac{S_3'}{u_2'-u_1}\Big)d\nu = 0,$$

$$\frac{1-\beta}{g}S_3 - (1-\mu\beta)\int_0^1 \frac{1}{\omega_2'}\frac{\partial}{\partial\nu}\Big(\frac{S_3'-S_3}{u_1'-u_1}\Big)d\nu - \int_0^1 \frac{1}{\omega_1'}\frac{\partial}{\partial\nu}\Big(\frac{S_1'}{u_1'-u_2}\Big)d\nu = 0. \qquad (19)$$

It is easy to verify that the functions

$$l_1^i = r\gamma_2^i(u_1-k_i)^{-1}, \qquad l_3^i = (1-\gamma_1^i)(u_2-k_i)^{-1} \qquad (20)$$

satisfy system (19). Therefore, the desired functions S_1 and S_2 can be written as

$$S_1 = S_1^* + r\sum_{i=1}^4 \frac{C_i\gamma_2^i}{u_1-k_i}, \qquad S_3 = S_3^* + \sum_{i=1}^4 \frac{C_i(1-\gamma_1^i)}{u_2-k_i}.$$

By choosing the quantities C_i independent of λ, we make S_1^* and S_3^* vanish for $\lambda = 0$ and $\lambda = 1$. After simple transformations, we obtain the following system of singular integral equations for the functions $\tilde{S}_1(u_1) = S_1^*(\lambda)$ and $\tilde{S}_3(u_2) = S_3^*(\lambda)$:

$$\frac{\mathrm{Re}\,(\chi^+(u_1))}{rg}\tilde{S}_1 - \frac{1+g\mu\chi_2(u_1)}{r}\int_{u_{10}}^{u_{11}}\Big(\frac{1}{\omega_1'}\Big)_{u_1'}\frac{\tilde{S}_1'\,du_1'}{u_1'-u_1} + \int_{u_{20}}^{u_{21}}\Big(\frac{1}{\omega_2'}\Big)_{u_2'}\frac{\tilde{S}_3'\,du_2'}{u_2'-u_1} = 0,$$

$$\frac{\mathrm{Re}\,(\chi^+(u_2))}{g}\tilde{S}_3 + \int_{u_{10}}^{u_{11}}\Big(\frac{1}{\omega_1'}\Big)_{u_1'}\frac{\tilde{S}_1'\,du_1'}{u_1'-u_2} + (1+\mu g\chi_1(u_2))\int_{u_{20}}^{u_{21}}\Big(\frac{1}{\omega_2'}\Big)_{u_2'}\frac{\tilde{S}_3'\,du_2'}{u_2'-u_2} = 0. \qquad (21)$$

The singular integral equations (21) contain both a characteristic part and a first-order Fredholm operator, which considerably complicates their solution.

If Eqs. (21) has only a trivial solution, the system of eigenfunctionals $\boldsymbol{F}^{i\lambda}$ and $\boldsymbol{F}^i$ $(i = 1, 2, 3, 4)$ is complete. Indeed, from the relations $(\boldsymbol{F}^i, \boldsymbol{S}) = 0$ we obtain a linear homogeneous system of four equations $\Gamma \boldsymbol{C} = 0$ for C_i. The components of the matrix Γ are given by the formula $\Gamma_{ij} = (\boldsymbol{F}^i, \boldsymbol{l}^j)$, where the vector function $\boldsymbol{l}^j$ has the components l_1^j and l_3^j given by formula (20); $l_2^j = \omega_1^{-1} l_{1\lambda}^j$ and $l_4^j = \omega_2^{-1} l_{3\lambda}^j$. For $i \neq j$,

$$\Gamma_{ij} = -\frac{r}{g(k_i - k_j)} \Big(\gamma_2^j (1 - \gamma_2^i)(\gamma_1^i - \gamma_1^j) + \gamma_1^i (1 - \gamma_1^j)(\gamma_2^i - \gamma_2^j)\Big) = 0$$

since, by virtue of the characteristic equation (12), the quantities γ_1^i and γ_2^i are linked by the relation $1 - \gamma_1^i - \gamma_2^i + \mu \gamma_1^i \gamma_2^i = 0$. The diagonal components of the matrix Γ are different from zero:

$$\Gamma_{ii} = \frac{r \gamma_2^i (1 - \gamma_2^i)}{g(\mu \gamma_2^i - 1)} \, \chi'(k_i) \neq 0.$$

Thus, the problem of the hyperbolicity of the two-layer rotational shallow-water equations (5) reduces to an analysis of the unique solvability of the singular integral equations (21).

3. Case of a Weak Density Discontinuity. We consider the situation where fluid density stratification is nearly absent $\rho_1 \to \rho_2$ $(r \to 1$ and $\mu \to 0)$. Passage to the limit in Eqs. (5) yields a simpler model for a homogeneous fluid with a slip of the layers. In this case, the characteristic equation has the form

$$\chi(k) = 1 - g \int_0^1 \frac{H_1 \, d\lambda}{(u_1 - k)^2} - g \int_0^1 \frac{H_2 \, d\lambda}{(u_2 - k)^2} = 0. \tag{22}$$

Equation (22) has four real roots if the condition $\chi(k_*) > 0$ $[k_* = (u_{11} + u_{20})/2]$ or its equivalent condition 5 in (15) is satisfied. The system of characteristic relations (18) reduces to the Riemann integral invariants

$$R_j = u_j - g \int_0^1 \frac{H_1' \, d\nu}{u_1' - u_j} - g \int_0^1 \frac{H_2' \, d\nu}{u_2' - u_j}, \qquad \omega_j = u_{j\lambda} H_j^{-1} \qquad (j = 1, 2),$$

$$r_i = k_i - g \int_0^1 \frac{H_1' \, d\nu}{u_1' - k_i} - g \int_0^1 \frac{H_2' \, d\nu}{u_2' - k_i} \qquad (i = 1, 2, 3, 4)$$

conserved on the characteristics

$$(\partial_t + u_j \partial_x) R_j = 0, \qquad (\partial_t + u_j \partial_x) \omega_j = 0, \qquad (\partial_t + k_i \partial_x) r_i = 0.$$

In addition, system (21) for $\mu = 0$ is represented as a homogeneous singular integral equation, adjoint to the characteristic equation, on the discontinuous contours [12]

$$A(\xi) S(\xi) - \frac{1}{\pi i} \int_L \frac{B(\xi') S(\xi') \, d\xi'}{\xi' - \xi} = 0. \tag{23}$$

Here $A = \operatorname{Re}(\chi^+)$, $B = \operatorname{Im}(\chi^+)$ and $S(\xi) = \tilde{S}_1$ if $\xi \in [u_{10}, u_{11}]$, and $S(\xi) = \tilde{S}_3$ if $\xi \in [u_{20}, u_{21}]$, L is a discontinuous line consisting of the segments $[u_{10}, u_{11}]$ and $[u_{20}, u_{21}]$, and χ^+ are the limiting values of the complex function

$$\chi(z) = 1 + g \sum_{j=1}^{2} \Big(\frac{1}{\omega_{j1}(u_{j1} - z)} - \frac{1}{\omega_{j0}(u_{j0} - z)} - \int_{u_{j0}}^{u_{j1}} \Big(\frac{1}{\omega_j'}\Big)_{u_j'} \frac{du_j'}{u_j' - z} \Big)$$

from the upper half plane on the real axis.

The introduction the function

$$\Psi(z) = \frac{1}{2\pi i} \int_L \frac{B(\xi) S(\xi) \, d\xi}{\xi - z}$$

reduces the solution of Eq. (23) to the solution of the homogeneous conjugation problem (Riemann problem)

$$\Psi^+(z) = G(z) \Psi^-(z), \qquad G(z) = \chi^+(z)/\chi^-(z) \tag{24}$$

for determining the analytic complex function $\Psi(z)$ from the boundary conditions on the discontinuous line L.

If the index of the function χ is equal to zero, the conjugation problem (24) has only a trivial solution in the class of functions vanishing at infinity [12]. If condition 5 in (15) and conditions (16) are satisfied, the index of the conjugation problem is equal to zero, and, hence, $S \equiv 0$. Thus, the completeness of the system of eigenfunctionals is proved and the hyperbolicity of the model (5) for $r = 1$ is established.

4. Case of a Strong Density Discontinuity. Another limiting case is strong stratification, i.e., the density in the lower layer ρ_1 is considerably higher than the density in the upper layer ρ_2, whose thickness is small. Setting $r = 0$ in Eqs. (5), we obtain a simplified model for a two-layer fluid. Obviously, the equations of motion split: in each of the layers, the fluid flow is described by a single-layer model of rotational shallow water [11]. In this case, the interface between the fluids, specified by the equation $y = h_1(t, x)$, is formed only under the influence of the heavy fluid in the lower layer. If the functions u_1 and H_1 are found from the first two equations of system (5) for $r = 0$, then the functions u_2 and H_2 are obtained by solving a similar problem with the right side corresponding to the case where the "bottom" $[y = h_1(t, x)]$ for the upper layer changes under the specified law. In [11], it was shown that the single-layer rotational shallow-water equations reduce to Riemann invariants and the model is generalized-hyperbolic for flows with a monotonic velocity profile over the depth if the conditions

$$\Delta \arg \frac{\chi^+(u)}{\chi^-(u)} = 0, \qquad \chi^{\pm} \neq 0$$

are satisfied for the limiting values of the analytic function $\chi(z) = \int_0^1 (u' - z)^{-2} H' \, d\nu$ on the segment $[u(t, x, 0), u(t, x, 1)]$.

Conclusions. The propagation of long-wave perturbations in a two-layer stratified rotational fluid was studied theoretically. The velocities of perturbation propagation were determined and the conditions of the absence of complex characteristic roots, necessary for the hyperbolicity of the model, were formulated. The characteristic form of the integrodifferential equations was calculated. The problem of the equivalence of the initial system and the characteristic form was reduced to studying the solvability of singular integral equations. In the limiting cases of strong and weak stratifications, the generalized hyperbolicity of the models was established and the existence of Riemann integral invariants was shown.

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